

Decomposition of Graphs: Strongly Connected Components

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Understand the definition of a strongly connected component of a directed graph.
- Give some other notions of connectivity within a directed graph.

Outline

1 Motivation

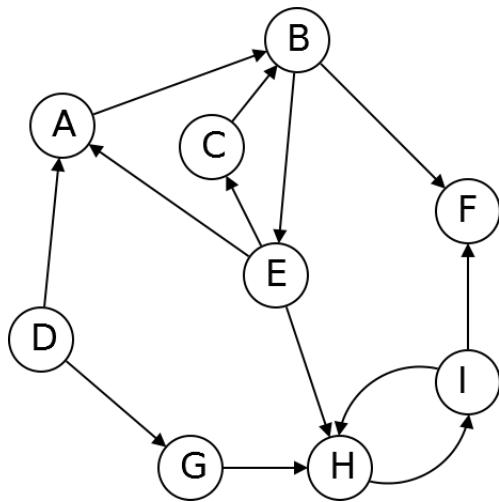
2 Definition

Connectivity in Digraphs

In undirected graphs, have connected components.

Directed graphs are more complicated.

Example



Possible Notions

- Connected by edges in any direction.
- One vertex reachable from another.
- Two vertices both reachable from the other.

Outline

1 Motivation

2 Definition

Strongly Connected Components

Definition

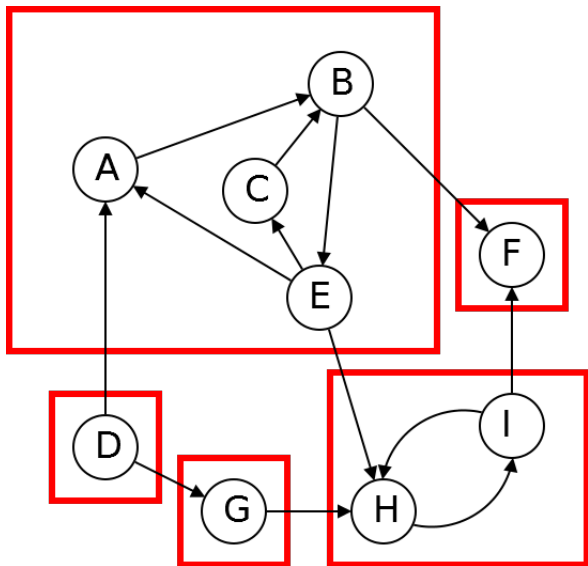
Two vertices v , w in a directed graph are **connected** if you can reach v from w and can reach w from v .

Result

Theorem

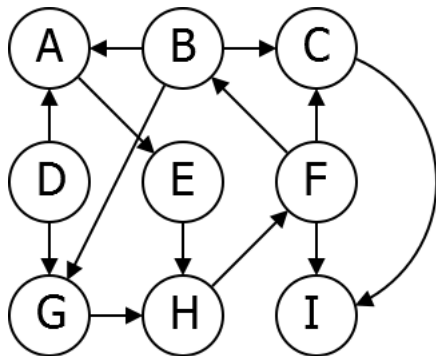
A directed graph can be partitioned into **strongly connected components** where two vertices are connected if and only if they are in the same component.

Example



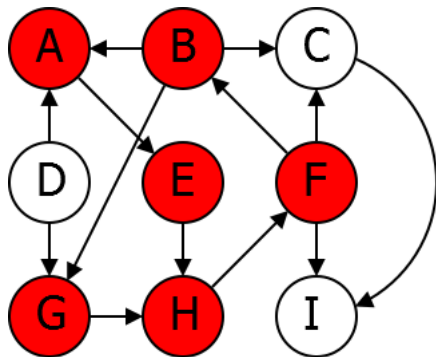
Problem

What is the SCC of A ?



Solution

A, B, E, F, G, H.



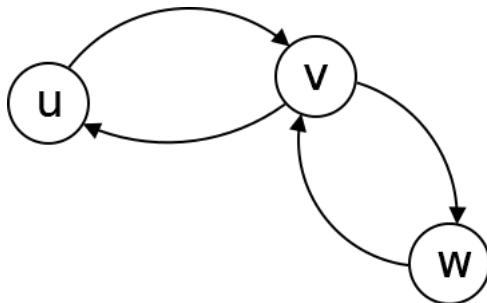
Result

Theorem

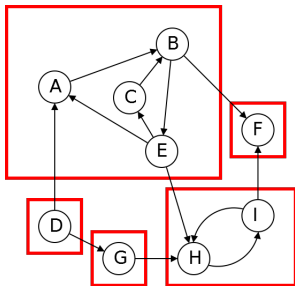
A directed graph can be partitioned into **strongly connected components** where two vertices are connected if and only if they are in the same component.

Proof

Need to show an equivalence relation.

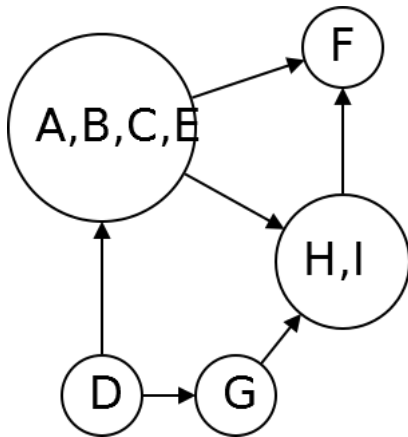


Metagraph



We can also draw a **metagraph** showing how the strongly connected components connect to one another.

Example



Note: It's a DAG.

DAG

Theorem

The metagraph of a graph G is always a DAG.

Proof

Proof.

Suppose not. Must be a cycle \mathcal{C} . Any nodes in cycle can reach any others. Should all be in same SCC. Contradiction. □

Summary

- Can partition vertices into strongly connected components.
- Metagraph describes how strongly connected components connect to each other.
- Metagraph always a DAG.

Next Time

How to compute the strongly connected components of a graph.