

# Linear Programming: Convex Polytopes

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# Learning Objectives

- Understand what a convex polytope is and why it is relevant to linear programming.
- Get a feel for what a convex polytope looks like.
- Prove some basic facts about convex polytopes.

# Linear Programs

Optimize linear function given linear inequality constraints.

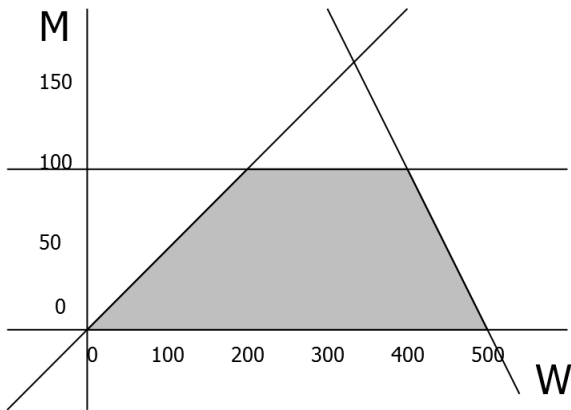
# Linear Programs

Optimize linear function given linear inequality constraints.

Want to understand region of points defined by inequalities.

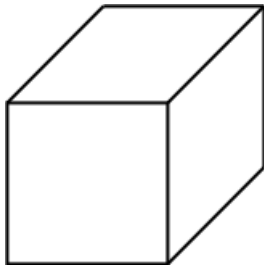
# Example

From factory example



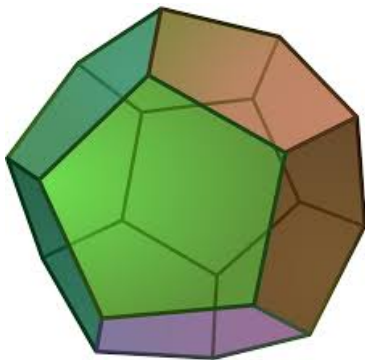
## Example II

Equations:  $0 \leq x, y, z \leq 1$  give cube.



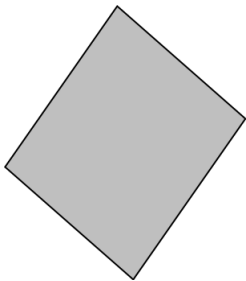
# In General

Get what's called a **convex polytope**



# Hyperplanes

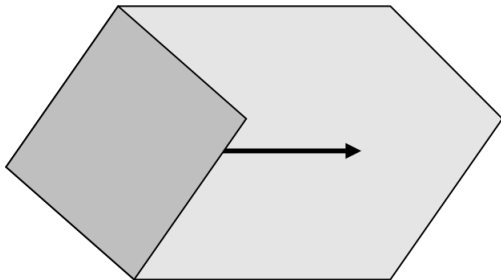
A single linear equation defines a hyperplane.





# Hyperplanes

A single linear equation defines a hyperplane.



An inequality, defines a halfspace.

# Polytopes

So a **system** of linear inequalities, defines a region bounded by a bunch of hyperplanes.

## Definition

A **polytope** is a region in  $\mathbb{R}^n$  bounded by finitely many flat surfaces.

# Polytopes

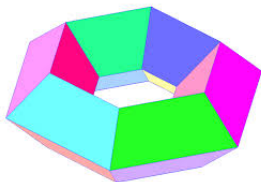
So a **system** of linear inequalities, defines a region bounded by a bunch of hyperplanes.

## Definition

A **polytope** is a region in  $\mathbb{R}^n$  bounded by finitely many flat surfaces. These surfaces may intersect in lower dimensional **facets** (like edges), with zero-dimensional facets called **vertices**.

# More Conditions

But not **every** polytope is possible.

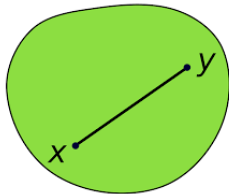


Everything must be on **one side** of each face.

# Convexity

## Definition

A region  $\mathcal{C} \subset \mathbb{R}^n$  is **convex**, if for any  $x, y \in \mathcal{C}$ , the line segment connecting  $x$  and  $y$  is contained in  $\mathcal{C}$ .



# Convexity

## Lemma

An intersection of halfspaces is convex.

# Proof

- Defined by  $Ax \geq b$ .
- Need for  $x, y \in \mathcal{C}$  and  $t \in [0, 1]$ ,  
 $tx + (1 - t)y \in \mathcal{C}$ .

$$\begin{aligned} A(tx + (1 - t)y) &= tAx + (1 - t)Ay \\ &\geq tb + (1 - t)b \\ &= b. \end{aligned}$$

# Convex Polytope

## Theorem

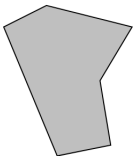
The region defined by a system of linear inequalities is always a convex polytope.



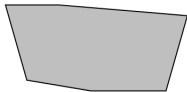
# Problem

Which of these figures is a convex polytope?

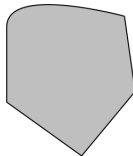
A



B



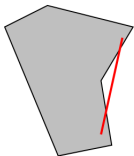
C



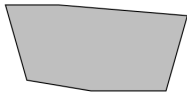
# Solution

Only B.

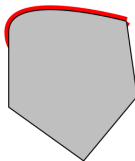
A



B



C



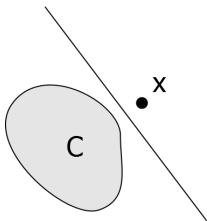
# Lemmas

We will conclude with a couple important lemmas about convex polytopes.

# Separation

## Lemma

Let  $\mathcal{C}$  be a convex region and  $x \notin \mathcal{C}$  a point. Then there is a hyperplane  $H$  separating  $x$  from  $\mathcal{C}$ .



# Separation

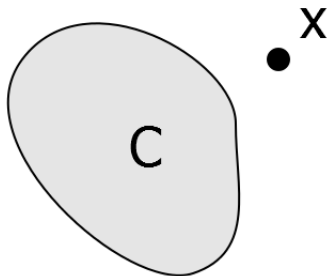
## Lemma

Let  $\mathcal{C}$  be a convex region and  $x \notin \mathcal{C}$  a point. Then there is a hyperplane  $H$  separating  $x$  from  $\mathcal{C}$ .

Note that if  $\mathcal{C}$  is given by a system of linear inequalities, we can just find one of the defining inequalities that  $x$  violates.

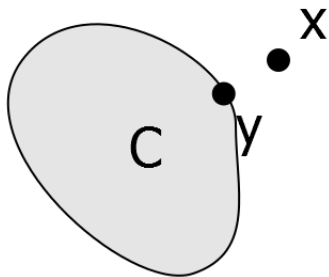
# Proof (Optional)

Start with  $x$ .



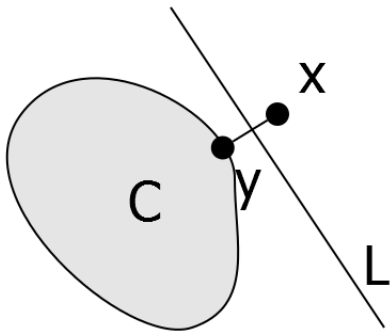
# Proof (Optional)

Let  $y$  be closest point in  $\mathcal{C}$



# Proof (Optional)

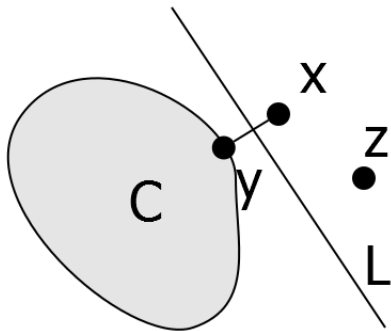
Let  $L$  be the perpendicular bisector of  $xy$ .





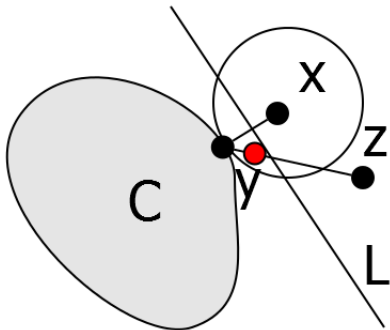
# Proof (Optional)

If  $z \in \mathcal{C}$  on wrong side of  $L$ ,



# Proof (Optional)

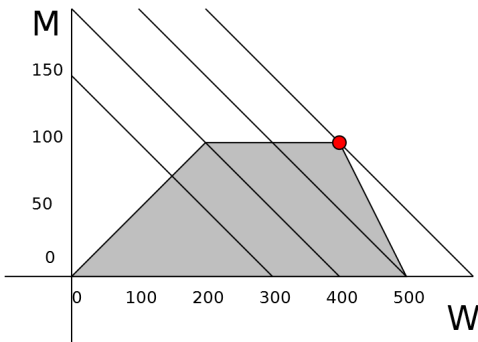
$yz$  contains point closer to  $x$ . Contradiction.



# Extreme Points

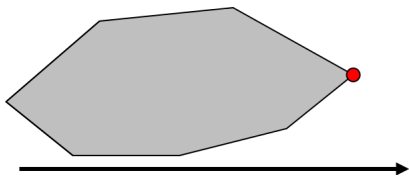
## Lemma

A linear function on a polytope takes its minimum/maximum values on vertices.



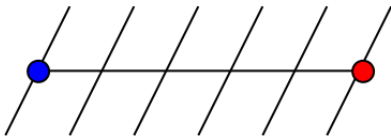
# Intuition

The corners are the only extreme points.  
Optima must be there.



# Idea

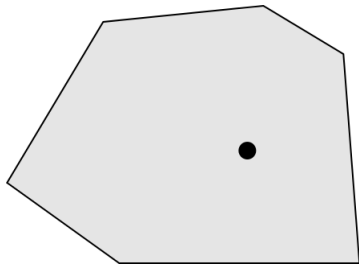
Linear function on segment takes extreme values on ends.



Use to push towards corners.

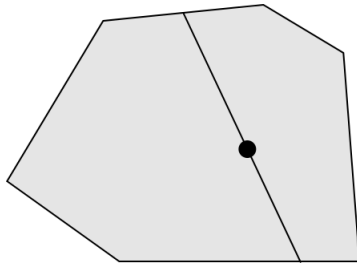
# Proof

Start at any point.



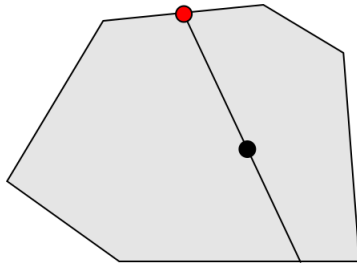
# Proof

Pick line through point.



# Proof

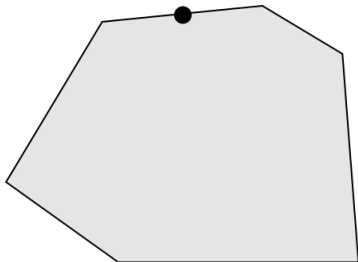
Extreme values at endpoint.





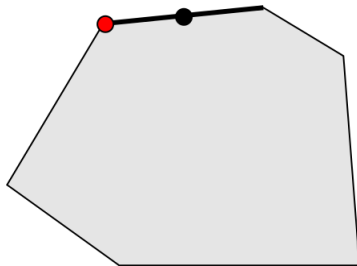
# Proof

Point is on a facet.



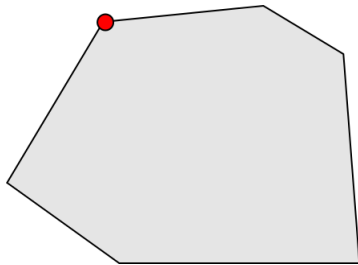
# Proof

Repeat to push to lower dimensional facet.



# Proof

Eventually on a vertex.



# Summary

- Region determined by LP always convex polytope.
- Optimum always at vertex.
- Can separate from outside points by hyperplanes.